Strategy Lines and Optimal Mixed Strategy for R

Best counterstrategy for C for given mixed strategy by R

In the previous lecture we saw that if R plays a particular mixed strategy, $[p, 1-p]$, and shows no intention of changing it, the expected payoff for R (and hence C) varies when C varies his/her strategy. Obviously C wants to choose the strategy that keeps R's expected payoff at a minimum. In the last lecture, we compared specific counterstrategies for C and chose the best. However C has infinitely many options for a counterstrategy and a pairwise comparison of all is not possible. In order to determine the **best counterstrategy for** C, we take a closer look at how the expected payoff for R varies as C varies his/her counterstrategy given that R continues to play a given strategy $[p, 1-p]$. As with the minimax method for strictly determined games, this will ultimately lead to a determination of the best strategy for R.

Example Lets go back to an earlier example of a zero-sum game with pay-off matrix for R given by

$$\begin{bmatrix} -1 & 3 \\ 2 & -2 \end{bmatrix}$$

and lets assume that R plays $\begin{bmatrix} .8 & .2 \end{bmatrix}$. Lets also assume that R is showing no signs of changing his/her strategy and C is exploring his/her options. C's goal is to minimize R's expected payoff, thus maximizing his/her own.

First let's consider **what happens when C plays a pure strategy**.

We will start with pure strategy $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ where C **always plays column 1**. The expected payoff for R if C plays column 1 only is

$$\begin{bmatrix} .8 & .2 \end{bmatrix} \begin{bmatrix} -1 & 3 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} .8 & .2 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = [.8(-1) + .2(2)] = [-.8 + .4] = [-.4]$$

On the other hand, if C plays the pure strategy $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ (where C **always plays column 2**), the expected payoff for R is

$$\begin{bmatrix} .8 & .2 \end{bmatrix} \begin{bmatrix} -1 & 3 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} .8 & .2 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \end{bmatrix} = [.8(3) + .2(-2)] = [2.4 - .4] = [2].$$

Thus, of these two strategies, **assuming that R continues to play the strategy $\begin{bmatrix} .8 & .2 \end{bmatrix}$, the better one for C is the strategy** $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, giving an expected payoff of .4 for C.

Claim If C chooses any other mixed strategy, $\begin{bmatrix} q \\ (1-q) \end{bmatrix}, 0 \le q \le 1$, the expected payoff for R will be a number between $-.4$ and 2.

Proof of claim : If C uses the strategy $\begin{bmatrix} q \\ (1-q) \end{bmatrix}$, the expected payoff for R will be

$$\begin{bmatrix} .8 & .2 \end{bmatrix} \begin{bmatrix} -1 & 3 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} q \\ 1-q \end{bmatrix} = \begin{bmatrix} -.4 & 2 \end{bmatrix} \begin{bmatrix} q \\ 1-q \end{bmatrix} = [-.4(q) + .2(1-q)]$$

We have $2 > -.4$, therefore $[-.4(q) + .2(1-q) > -.4(q) + (-.4)(1-q) = -.4$.
We also have $2 = 2q + 2(1-q) > -.4q + 2(1-q)$.

Conclusion: If R continues to play the strategy $\begin{bmatrix} .8 & .2 \end{bmatrix}$, the best counterstrategy for C is a pure strategy; always playing the column that minimizes R's payoff.

Applying the same reasoning as that in the example above, we can make similar conclusions for the general case:

General Case: Suppose R plays any mixed strategy $[p, 1-p]$, with a payoff matrix given by

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

Then if C plays a pure strategy $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ or $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, the payoff for R will be

$$\begin{bmatrix} p & (1-p) \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = [pa_{11} + (1-p)a_{21}] \text{ or}$$

$$\begin{bmatrix} p & (1-p) \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = [pa_{12} + (1-p)a_{22}] \text{ respectively.}$$

It follows, as above, that the expected payoff for R for any mixed strategy that C might choose will be between these two numbers.

> If R always plays the strategy $\begin{bmatrix} p & (1-p) \end{bmatrix}$, the best counterstrategy for C is a pure strategy; always playing the column that minimizes R's payoff. Similarly, if C always plays the strategy $\begin{bmatrix} q \\ (1-q) \end{bmatrix}$, the best counterstrategy for R will be a pure strategy, where R always plays the row which will maximize R's expected payoff.

Example Recall the pay-off matrix for General Roadrunner:

		C. attacks	
		B	S
R.	B	80%	100%
places bomb	S	90%	50%

If General Roadrunner plays a strategy of $(.3, .7)$, what counterstrategy should general Coyote play in order to minimize the number of bombs which reach their target?

Example: Should I have the operation? Some surgeries are essential for certain medical conditions but come with such a risk that it might be best not to undertake them if they can be avoided. It may not be easy to diagnose these conditions and often patients know only a probability that they have the medical condition. Let us consider a simple example where someone who is contemplating a risky surgery for a serious disease given that their doctor says there is a 50% chance that they have the disease. Here the opponent is Nature and we will make Nature the Column player. The Patient will be the Row player and the payoff is given in years of life expectancy for each of the four situations, where D and ND denote having and not having the disease respectively and S and NS denote a decision to have or not have surgery respectively. The payoff matrix is given below

		Nature	
		D	ND
Patient	S	15	30
	NS	2	40

Using our game theory model here, we can say that Nature's strategy is $\begin{bmatrix} .5 \\ .5 \end{bmatrix}$ and the patient want to find the best counterstrategy. In other words, is it better for the patient to have the surgery or not? Calculate the expected payoff (in life expectancy) for R for both decisions.

Strategy Lines and R's optimal mixed strategy

Given a $2 \times n$ payoff matrix, we can draw a picture of the possible payoffs for R as shown in the example below. We draw lines representing R's payoff for each of C's pure strategies. (This payoff will vary as p varies in R's strategy $[p, 1-p]$.) These lines are called strategy lines.

Example Lets look at the above example again, where the payoff matrix is given by

$$\begin{bmatrix} -1 & 3 \\ 2 & -2 \end{bmatrix}.$$

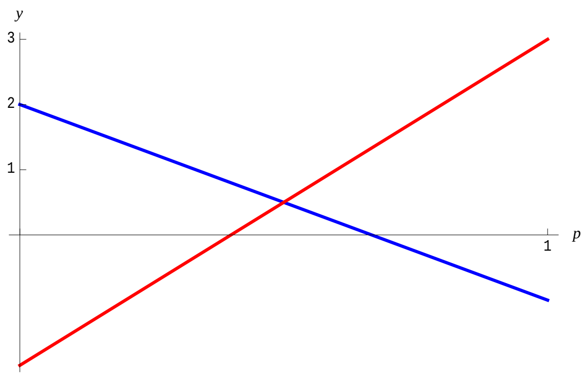
Let $[p, 1-p]$ denote R's strategy. We draw a co-ordinate system with the variable p on the horizontal axis and $y =$ the expected payoff for R on the vertical axis. **The lines shown give the expected payoff for R for the two pure strategies that C might pursue. These are called strategy lines.**

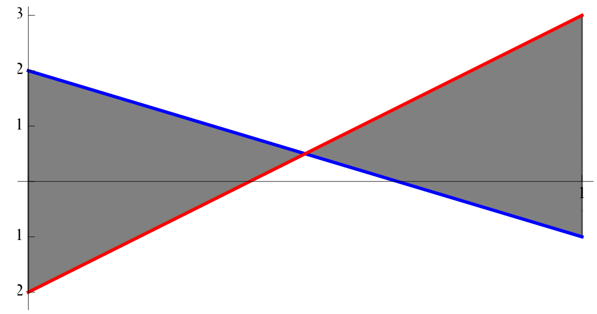
If R plays $[p, 1-p]$ and C plays $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, the expected payoff for R is

$$\begin{bmatrix} p & (1-p) \end{bmatrix} \begin{bmatrix} -1 & 3 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -p + 2(1-p) & 3p - 2(1-p) \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

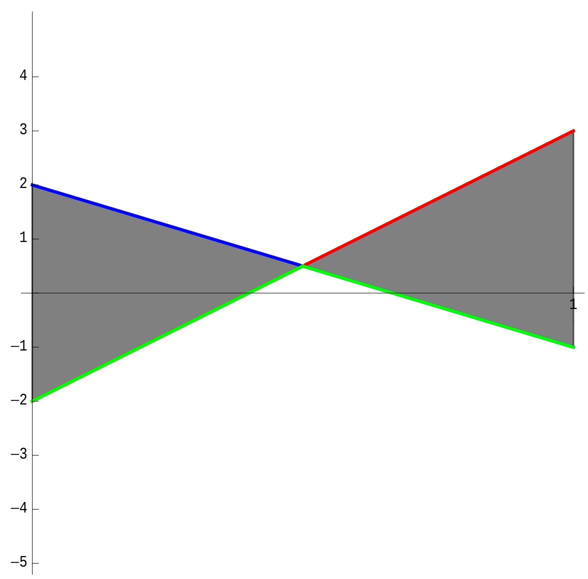
$$= \left[\begin{array}{cc} 2-3p & 5p-2 \end{array} \right] \left[\begin{array}{c} 1 \\ 0 \end{array} \right] = [2-3p]$$

Thus the line showing the expected value for R when C always plays Col. 1 is $\boxed{y = 2 - 3p}$ (shown in blue below).
Similarly, we see that the line showing the expected value for R when C always plays Col. 2 is $\boxed{y = 5p - 2}$ (shown in red below).





Now R can choose the value of p and we assume that C will respond appropriately and choose their best possible counterstrategy. Recall that whatever strategy C chooses, R's expected value will be in the shaded region on the right. The best counterstrategy for C is to choose the pure strategy that will give the minimum expected payoff for R. Hence, **if C chooses the best counterstrategy for any given choice of p made by R, R's payoff will be minimized and will appear along the line highlighted in green below.**



Now since R wants to maximize his/her payoff,R **chooses the strategy corresponding to the value of p which gives the maximum along the green line. This is where the lines in this picture meet.**

Find the value of p for which the above strategy lines meet and find R's best mixed strategy.

Example Recall the pay-off matrix for General Roadrunner:

		C. attacks	
		B	S
R.	B	80%	100%
places bomb	S	90%	50%

(a) Draw the strategy lines for C for this game.

(b) Use the strategy lines above to determine the optimal mixed strategy for General Roadrunner.

Note that this method can be used in situations where the payoff matrix with dimensions $2 \times n$.

Example In Tennis the server can serve to the forehand or serve to the backhand. The opponent can make a guess as to the type of serve and prepare for that serve or not guess at all. The payoff matrix for two players, Roger and Carlos, shown below shows the percentage of points ultimately won by the server in each situation.

		Carlos		
		Guess Forehand	**Guess Backhand**	**No Guess**
Roger	**Serve To Forehand**	40	70	45
	Serve To Backhand	80	60	65

(a) Does this payoff matrix have a saddle point?

(b) Plot Carlos' three strategy lines and highlight the lowest path.

(c) Determine the optimal mixed strategy for Roger for this game.

C's optimal strategy for an $n \times 2$ payoff matrix in a zero sum or constant sum game.

In the above examples we have found R's best mixed strategy keeping in mind that C will respond with the optimal counterstrategy. We saw that the analysis led to a solution where the payoff for R was the same no matter what strategy C chooses. The dynamic however does not necessarily end there. If C is not playing optimally and R can increase his/her payoff by responding with their best counterstrategy, we can assume that they will do so. So in order to find an equilibrium, we must also find the optimal strategy for C, which is a similar problem to that of finding the optimal strategy for R.

If C has only two options and the payoff matrix does not have a saddle point, then we can determine C's optimal mixed strategy in a manner similar to that shown above for R's optimal strategy. Namely, if C's strategy is denoted by $\begin{bmatrix} q \\ 1-q \end{bmatrix}$, we plot strategy lines for R in a Cartesian plane with horizontal axis q and vertical axis y = expected payoff for R. We then highlight the line corresponding to the maximum payoff for R. C will then choose the strategy which will minimize R's payoff.

Warning In previous lectures, we reformulated the linear programming problem in a different way for 2×2 zero sum games with positive entries. As a result there are many old exam questions of the following type:

1 Carlos (C) and Rosita (R) play a zero-sum game, with payoff matrix for Rosita given by

	C_1	C_2
R_1	10	3
R_2	5	9

If Rosita wants to find her optimal mixed strategy, given that Carlos always plays the best counterstrtaegy, which of the following linear programming problems must she solve?

$$\begin{array}{cc} \text{minimize} & x+y \\ \text{subject to} & \\ x \geq 0, & y \geq 0 \\ 10x+5y & \geq 1 \\ 3x+9y & \geq 1 \end{array} \qquad \begin{array}{cc} \text{minimize} & x+y \\ \text{subject to} & \\ x \geq 0, & y \geq 0 \\ 10x+3y & \leq 1 \\ 5x+9y & \leq 1 \end{array} \qquad \begin{array}{cc} \text{minimize} & x+y \\ \text{subject to} & \\ x \geq 0, & y \geq 0 \\ 10x+5y & \leq 1 \\ 3x+9y & \leq 1 \end{array} \qquad \begin{array}{cc} \text{maximize} & x+y \\ \text{subject to} & \\ x \geq 0, & y \geq 0 \\ 10x+3y & \geq 1 \\ 5x+9y & \geq 1 \end{array} \qquad \begin{array}{cc} \text{minimize} & x+y \\ \text{subject to} & \\ x \geq 0, & y \geq 0 \\ 10x+3y & \geq 1 \\ 5x+9y & \geq 1 \end{array}$$

and

2 Connor (C) and Rosemary (R) play a zero-sum game, with payoff matrix for Rosemary given by

	C_1	C_2
R_1	4	3
R_2	2	5

If the solution to the linear programming problem:

$$\begin{array}{ccc} \text{minimize} & x+y & \\ \text{constraints} & x \geq 0, & y \geq 0 \\ & 4x+2y & \geq 1 \\ & 3x+5y & \geq 1 \end{array}$$

is given by $x = \frac{3}{14}$, $y = \frac{1}{14}$, which of the following give the optimal strategy and ν = expected payoff for Rosemary?

(a) $\left(\frac{3}{4}, \frac{1}{4}\right)$, $\nu = \frac{7}{2}$ (b) $\left(\frac{1}{4}, \frac{3}{4}\right)$, $\nu = \frac{7}{2}$ (c) $\left(\frac{1}{2}, \frac{1}{2}\right)$, $\nu = \frac{7}{2}$ (d) $\left(\frac{1}{2}, \frac{1}{2}\right)$, $\nu = \frac{2}{7}$ (e) $\left(\frac{1}{4}, \frac{3}{4}\right)$, $\nu = \frac{2}{7}$

You should ignore these questions when reviewing for your exam.

Sample Exam Questions

1 Rocky (The Row player) and Creed (The Column Player) play a zero-sum game. The payoff matrix for Rocky, is given by:

$$\begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix}.$$

If Rocky plays the mixed strategy (.6 .4), which of the following mixed strategies should Creed play to maximize his (Creed's) expected payoff in the game?

(a) $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ (b) $\begin{bmatrix} .6 \\ .4 \end{bmatrix}$ (c) $\begin{bmatrix} .4 \\ .6 \end{bmatrix}$ (d) $\begin{bmatrix} .3 \\ .7 \end{bmatrix}$ (e) $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

2 Rose (R) and Colm (C) play a zero-sum game. The payoff matrix for Rose, is given by:

$$\begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}.$$

Which of the following give the strategy lines corresponding to the fixed strategies of Colm where Rose's strategy is given by $[p(1-p)]$ and Rose's expected payoff is denoted by y.

(a) $\begin{array}{c} y = p + 1 \\ y = 3 - 4p \end{array}$ (b) $\begin{array}{c} y = 3p - 1 \\ y = 3 - 2p \end{array}$ (c) $\begin{array}{c} y = 2p + 1 \\ y = 3p - 1 \end{array}$

(e) $\begin{array}{c} y = 2 - p \\ y = 4p - 1 \end{array}$ (e) $\begin{array}{c} y = 2 - 3p \\ y = 4p + 1 \end{array}$